ORA Mathematics, 1(1), e2026006
DOI: 10.XXXXX/oramath.e2026006
Received: 25 April 2026
Revised: 05 May 2026
Accepted: 07 May 2026
Published: 08 May 2026

RESEARCH ARTICLE

Bootstrap-calibrated estimation for progressively censored competing risks data under the Gompertz-Lindley model

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Abstract: This paper develops a Bootstrap-Calibrated Hybrid Estimator (BCHE) for the parameters of the Gompertz-Lindley distribution (GLD) under progressive Type-II censoring in the presence of competing risks. The proposed approach combines maximum likelihood estimation via Newton-Raphson iteration with a parametric bootstrap calibration loop to correct finite-sample bias in all three model parameters simultaneously. Bias-corrected and accelerated (BCa) bootstrap confidence intervals are constructed and compared against the standard Wald-type asymptotic confidence intervals obtained from the observed Fisher information matrix. A comprehensive Monte Carlo simulation study is conducted using the heart disease dataset parameters from the literature, covering four sample-size configurations and three progressive censoring schemes. Simulation results confirm that the BCHE substantially reduces relative absolute bias and mean squared error for the GLD shape parameter, where classical maximum likelihood estimation is known to perform poorly under moderate to heavy censoring. Bootstrap BCa intervals achieve empirical coverage closer to the nominal 95% level with shorter average lengths compared to Wald-type intervals. A real data application to heart disease patient records with two competing failure causes—myocardial infarction and death from other causes—further illustrates the practical utility of the proposed methodology and demonstrates the suitability of the GLD for competing risks lifetime data.

Keywords: Progressive Type-II censoring; competing risks modeling; Gompertz-Lindley lifetime model; bootstrap bias correction; BCa confidence intervals; statistical survival analysis

Mathematics Subject Classification: 62F40, 62N05, 62F12

1. Introduction

Survival analysis with competing risks arises when a subject is exposed to multiple mutually exclusive failure mechanisms and the occurrence of any one failure event prevents the observation of all remaining causes. This is a common situation in biomedical, engineering and actuarial problems. For example, in clinical studies of cardiovascular disease, competing risks such as myocardial infarction and death from other causes are modeled as different types of terminal events, and treating one as an ordinary censoring event with respect to the other leads to an overestimation of cause-specific risk estimates. In reliability engineering , a component manufactured by manufacturing may fail due to fatigue , corrosion or a manufacturing defect . Only the first mechanism to appear causes an observable failure . Cox [17] laid down

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the foundational formalism for competing risks inference. Since then the field has blossomed with rich parametric and nonparametric inferential tools for cause-specific hazard functions and cumulative incidence functions.

Progressive Type-II censoring (PC-II) is a common method in lifetime experiments where surviving units can be removed from the test intentionally during the test to reduce the cost or time of experimentation. In this scheme, a n units are put on test; a certain number, $m < n$, of failures are to be targeted and, at the time of the ith observed failure, a pre-specified number, R_i, of surviving units are randomly removed. The constraint to be satisfied is that of censoring $\sum_{i=1}^{m} R_i = n - m$. The scheme provides much experimental flexibility, including full samples ($R_i = 0$ for all i) and classical Type-II censoring ($R_i = 0$ for $i < m$, $R_m = n-m$) as special cases. The joint likelihood for the observed progressively censored data, coupled with competing risks, has a structured form that incorporates the cause-specific densities and survival functions at each observed failure time [2, 30].

The Gompertz-Lindley distribution (GLD) has a shape parameter α and two cause-specific scale parameters β_1 and β_2. It combines the monotone increasing hazard of the Gompertz family with the flexible tail behavior of the Lindley distribution. The resulting composite model is flexible to increasing, decreasing and unimodal hazard shapes making it applicable to heterogeneous lifetime populations in both biostatistical and engineering settings [24, 38]. Classical maximum likelihood estimators (MLE) for the GLD parameters are found by solving nonlinear score equations using Newton-Raphson iteration. MLEs in the GLD competing risks model are known to be consistent and asymptotically efficient with non-negligible finite-sample bias especially for the shape parameter α especially under heavier progressive censoring schemes and small to moderate sample sizes.

Bias correction of parametric estimators has been a longstanding issue in the statistical literature. Closed-form bias corrections for inference based on second-order expansions of the score function are available, but they require tractable higher-order derivatives of the log-likelihood, which are not available in closed form for the GLD. Resampling based corrections avoid this requirement by empirically estimating the bias from bootstrap replications of the fitted model. Parametric bootstrap methods sample directly from the fitted distribution and so are well suited to complex censored data settings. The bias-corrected and accelerated (BCa) interval [10, 13] corrects for both skewness of the bootstrap distribution and acceleration of the standard error, resulting in intervals with better coverage than percentile or standard bootstrap methods when the sampling distributions are not normal. Similar resampling developments for censored and progressively censored data are offered in [4, 25, 29, 31, 35, 36].

Bias correction methods for parametric models are a broad class of methods that have been studied in a variety of settings. Tsai et al. [33] proposed analytical bias corrections for the log-power-normal distribution, and showed significant reductions in mean squared error (MSE). Xin et al. [37] extended similar ideas to the unit exponential distribution and derived closed-form adjusted estimators and compared them to bootstrap estimators. Lemonte [21] explored bias corrections for the two-sided power distribution and showed that resampling-based methods work well when analytic formulas are not available. In panel and econometric models, bias correction strategies based on half-panel jackknife and split-sample methods were studied by Hahn et al. [15] and Ghanem [14], while Liu [22] investigated small-sample bias of Cohen's d and derived closed-form corrections. Tibbe and Montoya [32] showed that the standard BCa procedure tends to overcorrect systematically in the mediation analysis context and they proposed recalibrated acceleration constants. Their work provides useful context for BCa application in non-standard settings. Dai et al. [7] and Walsh and Jentsch [34] provide extensions of the wild bootstrap and block bootstrap that are relevant for spatial and time-series models. Elbayoumi and Mostafa [12] and Elbayoumi et al. [13] showed for bifurcating autoregressive models that the bootstrap bias correction can reduce estimator dispersion compared to uncorrected least squares. Goodness-of-fit procedures based on resampling methods are related to bootstrap calibration in Chandy et al. [6].

The motivation for the present study comes from several recent works in the literature of competing risks and progressive censoring. Alhidairah et al. [2] studied inference for generalized linear exponential competing risks models with progressive Type-II censoring and presented medical and industrial applications. The results obtained are benchmark results for comparing new estimators. Shi and Gui [30] estimated parameters of two Gompertz populations under the balanced joint progressive Type-II censoring and they gave directly comparable simulation evidence for Gompertz-family models. Singh et al. [31] investigated the inference for the balanced joint progressive Type-II censoring scheme. Xiang et al. [36] presented procedures for the inverse Weibull distribution under joint progressive censoring. Bera and Jana [5], Jana and Bera [18], Abo-Kasem et al. [1] and Qiao and Gui [27] provide further results on progressively censored reliability estimation. Alotaibi et al. [3] studied step-stress models for α-power Weibull lifetimes with progressive censoring, and Niu et al. [26] studied bathtub-shaped distributions under partially accelerated life testing with progressive censoring. Ranjbar [29] considered inference for x-gamma distribution under progressive Type-II censoring. Hakamipour [16] studied stress-strength reliability under copula dependent progressively censored samples. In [11] the results for bivariate generalized Rayleigh distributions under progressive censoring and random removal are presented. Bayesian and classical results for the Maxwell distribution under adaptive progressive Type-II censoring are given by Kumari et al. [20]. Dey et al. [8] investigated the Nadarajah-Haghighi distribution under constant-stress partially accelerated life tests with progressive censoring. Bayesian and Non-Bayesian Estimation for Bivariate Inverse Weibull Distributions," *Muhammed and Almetwally*, 2024. EL-Sagheer et al. [10] provided results on bootstrap confidence intervals and process capability indices with progressive censoring. Jeon et al. [19] suggested pivotal-based inference for the Pareto distribution under adaptive progressive Type-II censoring while Dhaene and Rosseel [9] explored resampling-based bias correction in structural equation models. Muralidharan and Bavagosai [24] provided instant failure analysis of Lindley distribution under progressive Type-II censoring. Hassan et al. [17] analysed progressive Type-II competing risks data with applications, providing the direct antecedent to the current framework.

The main contribution of the present paper is the development and evaluation of a Bootstrap-Calibrated Hybrid Estimator (BCHE) for the three GLD parameters $(\alpha, \beta_1, \beta_2)$ under progressive Type-II censoring with two competing failure causes. The estimator corrects the finite-sample MLE bias by means of a parametric bootstrap loop and constructs BCa confidence intervals which are closer to the nominal coverage level than the Wald-type asymptotic intervals. The proposed methodology is evaluated using a comprehensive simulation study using the parameters of the well-known heart disease dataset, and is demonstrated on real competing risks data from heart disease patients.

The remaining paper is organized as follows. In Section 2 we describe the competing risks model and the notation for progressive censoring. In Section 3 we develop the maximum likelihood estimators and asymptotic confidence intervals. Details of the Bootstrap-Calibrated Hybrid Estimator and construction of the BCa interval are given in Section 4. Section 5 presents the Monte Carlo simulation results for the performance of the estimators for different sample sizes and censoring schemes. We apply the methodology to the heart disease data set in Section 6. We discuss concluding remarks and directions for future work in Section 7.

2. Model And Notations

2.1. Competing Risks Framework

Consider an experiment involving n units subject to two distinct, mutually exclusive causes of failure. For unit $i = 1, 2, \ldots, n$, define the latent failure times T_{i1} and T_{i2} corresponding to causes 1 and 2, respectively. The observed failure time is $T_i = \min\{T_{i1}, T_{i2}\}$, and the failure cause indicator is $\delta_i \in \{1, 2\}$. The latent times T_{i1} and T_{i2} are assumed independent and to follow the Gompertz-Lindley distribution (GLD) with a common shape parameter α and cause-specific scale parameters β_1 and β_2, written $T_{i1} \sim \mathrm{GLD}(\alpha, \beta_1)$ and $T_{i2} \sim \mathrm{GLD}(\alpha, \beta_2)$.

2.2. Gompertz-Lindley Distribution

The GLD with parameters $\alpha > 0$ and $\beta > 0$ has probability density function (PDF), survival function (SF), and hazard rate function (HRF) given respectively by

$$f(t; \alpha, \beta) = \frac{\alpha^2 \beta e^{\beta t}\left(e^{\beta t} + \alpha + 1\right)}{(\alpha + 1)\left(e^{\beta t} + \alpha - 1\right)^3}, \quad t > 0, \tag{1}$$

$$S(t; \alpha, \beta) = \frac{\alpha^2 \left(e^{\beta t} + \alpha\right)}{(\alpha + 1)\left(e^{\beta t} + \alpha - 1\right)^2}, \tag{2}$$

$$H(t; \alpha, \beta) = \frac{\beta e^{\beta t}\left(e^{\beta t} + \alpha + 1\right)}{\left(e^{\beta t} + \alpha\right)\left(e^{\beta t} + \alpha - 1\right)}. \tag{3}$$

The GLD was introduced by Ghitany et al. as a composite model combining Gompertz and Lindley structures, and it is capable of exhibiting increasing, decreasing, and unimodal hazard shapes depending on α. This flexibility makes the GLD attractive for both biostatistical and reliability applications [38].

2.3. Minimum Lifetime Distribution

Let $X_1 \sim \mathrm{GLD}(\alpha, \beta_1)$ and $X_2 \sim \mathrm{GLD}(\alpha, \beta_2)$ be independent. The survival function of the minimum $X = \min\{X_1, X_2\}$ is

$$G(x; \alpha, \beta_1, \beta_2) = \frac{\alpha^4 \left(e^{\beta_1 x} + \alpha\right)\left(e^{\beta_2 x} + \alpha\right)}{(\alpha + 1)^2 \left(e^{\beta_1 x} + \alpha - 1\right)^2 \left(e^{\beta_2 x} + \alpha - 1\right)^2}. \tag{4}$$

2.4. Progressive Type-II Censoring Scheme

Let n identical units be placed on test under a censoring scheme $\mathbf{R} = (R_1, R_2, \ldots, R_m)$, where m denotes the targeted number of observed failures and $\sum_{i=1}^{m} R_i = n - m$. At the moment of the i-th failure, R_i surviving units are withdrawn. The cause indicator $\delta_i \in \{1, 2\}$

identifies which failure mechanism terminated unit i. With $m_j = \sum_{i=1}^{m} \mathbf{1}(\delta_i = j)$ denoting the total number of failures attributed to cause j, we have $m_1 + m_2 = m$.

Three censoring schemes are studied throughout this paper:

- **Scheme 1** (early): $R_1 = n - m$, $R_i = 0$ for $i \geq 2$.
- **Scheme 2** (balanced): $R_i = 1$ for $i = 1, \dots, n - m$; $R_i = 0$ otherwise.
- **Scheme 3** (late): $R_i = 0$ for $i < m$, $R_m = n - m$.

Scheme 1 concentrates all censoring at the first failure, yielding the least available information. Scheme 3 has the highest number of observations for most of the experiment and usually gives the most stable estimates. Withdrawals are uniform in scheme 2, which is a practical compromise [4].

3. Maximum Likelihood Estimation

3.1. Likelihood Function And Score Equations

Based on the progressively censored competing risks observation vector $(t_1, \delta_1, R_1), \dots, (t_m, \delta_m, R_m)$, the likelihood function for $\boldsymbol{\theta} = (\alpha, \beta_1, \beta_2)^\top$ is

$$\begin{aligned} L(\boldsymbol{\theta}; \mathbf{t}) = \frac{\alpha^{4m} \beta_1^{m_1} \beta_2^{m_2}}{(1+\alpha)^{2m}} \prod_{i=1}^{m} & \left[\frac{e^{\beta_1 t_i}(\alpha + e^{\beta_1 t_i} + 1)(\alpha + e^{\beta_2 t_i})}{(\alpha+1)^2(\alpha + e^{\beta_1 t_i} - 1)^3(\alpha + e^{\beta_2 t_i} - 1)^2} \right]^{\mathbf{1}(\delta_i = 1)} \\ & \times \left[\frac{e^{\beta_2 t_i}(\alpha + e^{\beta_1 t_i})(\alpha + e^{\beta_2 t_i} + 1)}{(\alpha+1)^2(\alpha + e^{\beta_1 t_i} - 1)^2(\alpha + e^{\beta_2 t_i} - 1)^3} \right]^{\mathbf{1}(\delta_i = 2)} \\ & \times \left[\frac{(\alpha + e^{\beta_1 t_i})(\alpha + e^{\beta_2 t_i})}{(\alpha+1)^2(\alpha + e^{\beta_1 t_i} - 1)^2(\alpha + e^{\beta_2 t_i} - 1)^2} \right]^{R_i}. \end{aligned} \tag{5}$$

The corresponding log-likelihood is

$$\begin{aligned} \ell \propto & \sum_{i=1}^{m} R_i \left[\log \frac{\alpha^2(\alpha + e^{\beta_1 t_i})}{(\alpha+1)(\alpha + e^{\beta_1 t_i} - 1)^2} + \log \frac{\alpha^2(\alpha + e^{\beta_2 t_i})}{(\alpha+1)(\alpha + e^{\beta_2 t_i} - 1)^2} \right] \\ & + \sum_{i=1}^{m} \mathbf{1}(\delta_i = 1) \log \frac{\alpha^4 \beta_1 e^{\beta_1 t_i}(\alpha + e^{\beta_1 t_i} + 1)(\alpha + e^{\beta_2 t_i})}{(\alpha+1)^2(\alpha + e^{\beta_1 t_i} - 1)^3(\alpha + e^{\beta_2 t_i} - 1)^2} \\ & + \sum_{i=1}^{m} \mathbf{1}(\delta_i = 2) \log \frac{\alpha^4 \beta_2 e^{\beta_2 t_i}(\alpha + e^{\beta_1 t_i})(\alpha + e^{\beta_2 t_i} + 1)}{(\alpha+1)^2(\alpha + e^{\beta_1 t_i} - 1)^2(\alpha + e^{\beta_2 t_i} - 1)^3}. \end{aligned} \tag{6}$$

Differentiating ℓ with respect to α, β_1, and β_2 and equating to zero gives the score equations. For β_1:

$$\sum_{i=1}^{m} R_i \left[\frac{t_i e^{\beta_1 t_i}}{\alpha + e^{\beta_1 t_i}} - \frac{2 t_i e^{\beta_1 t_i}}{\alpha + e^{\beta_1 t_i} - 1} \right] + \sum_{i=1}^{m} \mathbf{1}(\delta_i = 1) \left[\frac{1}{\beta_1} - \frac{3 t_i e^{\beta_1 t_i}}{\alpha + e^{\beta_1 t_i} - 1} + \frac{t_i e^{\beta_1 t_i}}{\alpha + e^{\beta_1 t_i} + 1} \right] = 0. \tag{7}$$

An analogous equation holds for β_2 with the cause-2 indicator, and the score equation for α involves sums over all observations. Because these equations admit no closed-form solutions, the Newton-Raphson algorithm is applied with data-driven initial values obtained from a coarse grid search, ensuring stable convergence to the maximum likelihood estimates $\hat{\boldsymbol{\theta}} = (\hat{\alpha}, \hat{\beta}_1, \hat{\beta}_2)^\top$ [17].

3.2. Asymptotic Confidence Intervals

Under standard regularity conditions, the MLE $\hat{\boldsymbol{\theta}}$ is asymptotically normally distributed:

$$\sqrt{m}\,(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \xrightarrow{d} \mathcal{N}_3\big(\mathbf{0},\, \mathcal{I}^{-1}(\boldsymbol{\theta})\big)\,, \tag{8}$$

where $\mathcal{I}(\boldsymbol{\theta})$ is the Fisher information matrix. Because the second partial derivatives of ℓ are analytically intractable for the GLD, the Fisher information is approximated by the observed information matrix $\mathbf{J}(\hat{\boldsymbol{\theta}}) = -\partial^2\ell/\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^\top\big|_{\hat{\boldsymbol{\theta}}}$, computed numerically. The estimated variance-covariance matrix is $\hat{\boldsymbol{\Sigma}} = \mathbf{J}^{-1}(\hat{\boldsymbol{\theta}})$, and an asymptotic $(1-\gamma)\times 100\%$ confidence interval for θ_j is

$$\hat{\theta}_j \pm z_{\gamma/2}\sqrt{\hat{\Sigma}_{jj}}, \quad j = 1, 2, 3, \tag{9}$$

where $z_{\gamma/2}$ is the upper $\gamma/2$ quantile of the standard normal distribution [5, 19]. These Wald-type asymptotic confidence intervals (ACI) serve as the baseline against which the proposed bootstrap intervals are compared.

4. Bootstrap-Calibrated Hybrid Estimator

4.1. Motivation And Overview

The maximum likelihood estimator for the GLD shape parameter α under progressive Type-II censoring exhibits well-documented finite-sample bias, particularly under Scheme 1 (early censoring) and for small to moderate sample sizes. This bias persists even as n grows because the fraction of complete observations m/n may be held fixed. Analytical bias correction requires third-order derivatives of the log-likelihood of the GLD, which are algebraically unwieldy. A parametric bootstrap calibration loop offers a computationally feasible and model-consistent alternative: it approximates the bias of $\hat{\alpha}$, $\hat{\beta}_1$, and $\hat{\beta}_2$ empirically by resampling from the fitted GLD model and corrects each estimate accordingly.

The BCa bootstrap interval further improves on the standard percentile interval by accounting for the non-normality and skewness of the bootstrap distribution of each estimator. This is especially relevant for α, whose sampling distribution under the GLD is right-skewed in small samples, leading to systematic undercoverage by standard Wald or percentile intervals [10, 12, 32].

4.2. Parametric Bootstrap Calibration

Let $\hat{\boldsymbol{\theta}} = (\hat{\alpha}, \hat{\beta}_1, \hat{\beta}_2)^\top$ denote the MLEs obtained from the observed progressively censored competing risks sample of size (n, m) under censoring scheme $\mathbf{R}$. The Bootstrap-Calibrated Hybrid Estimator proceeds as follows.

1. Using $\hat{\boldsymbol{\theta}}$, simulate B independent parametric bootstrap samples of size (n, m) from the GLD competing risks model under the same censoring scheme $\mathbf{R}$. Each bootstrap sample $b = 1, \ldots, B$ is generated by drawing $T_{i1}^{(b)} \sim \mathrm{GLD}(\hat{\alpha}, \hat{\beta}_1)$ and $T_{i2}^{(b)} \sim \mathrm{GLD}(\hat{\alpha}, \hat{\beta}_2)$, setting $T_i^{(b)} = \min\{T_{i1}^{(b)}, T_{i2}^{(b)}\}$, recording the cause indicator, and applying the progressive censoring scheme $\mathbf{R}$.

2. Compute the MLE $\hat{\boldsymbol{\theta}}^{(b)} = (\hat{\alpha}^{(b)}, \hat{\beta}_1^{(b)}, \hat{\beta}_2^{(b)})^\top$ from each bootstrap sample by the same Newton-Raphson procedure.

3. Estimate the bias of each parameter by

$$\widehat{\text{Bias}}(\hat{\theta}_j) = \frac{1}{B}\sum_{b=1}^{B}\hat{\theta}_j^{(b)} - \hat{\theta}_j, \quad j = 1, 2, 3. \tag{10}$$

4. Construct the bias-corrected point estimator

$$\tilde{\theta}_j^{\text{BC}} = \hat{\theta}_j - \widehat{\text{Bias}}(\hat{\theta}_j) = 2\hat{\theta}_j - \frac{1}{B}\sum_{b=1}^{B}\hat{\theta}_j^{(b)}. \tag{11}$$

The Bootstrap-Calibrated Hybrid Estimator (BCHE) for $\boldsymbol{\theta}$ is the triple $\tilde{\boldsymbol{\theta}}^{\text{BC}} = (\tilde{\alpha}^{\text{BC}}, \tilde{\beta}_1^{\text{BC}}, \tilde{\beta}_2^{\text{BC}})^{\top}$. The correction in Equation (11) is the standard jackknife-analogue formula for parametric bootstrap bias correction, and it is consistent under mild regularity conditions [13, 33, 37].

4.3. BCa Bootstrap Confidence Intervals

The bias-corrected and accelerated (BCa) bootstrap interval for θ_j uses two calibration constants: the bias-correction constant $\hat{z}_0$ and the acceleration constant $\hat{a}$. Let $\hat{\theta}_j^{(1)} \le \cdots \le \hat{\theta}_j^{(B)}$ denote the ordered bootstrap estimates. Then

$$\hat{z}_0 = \Phi^{-1}\left(\frac{\#\{\hat{\theta}_j^{(b)} < \hat{\theta}_j\}}{B}\right), \tag{12}$$

where Φ is the standard normal CDF. The acceleration constant is estimated by the jackknife:

$$\hat{a} = \frac{\sum_{k=1}^{m}(\bar{\varphi} - \hat{\varphi}_{(k)})^3}{6\left[\sum_{k=1}^{m}(\bar{\varphi} - \hat{\varphi}_{(k)})^2\right]^{3/2}}, \tag{13}$$

where $\hat{\varphi}_{(k)}$ is the MLE of θ_j with the k-th observation deleted and $\bar{\varphi} = m^{-1}\sum_k \hat{\varphi}_{(k)}$. The BCa interval at nominal level $1-\gamma$ is

$$\left(\hat{\theta}_j^{(\alpha_1 B)},\ \hat{\theta}_j^{(\alpha_2 B)}\right), \tag{14}$$

where

$$\alpha_1 = \Phi\left(\hat{z}_0 + \frac{\hat{z}_0 + z_{\gamma/2}}{1 - \hat{a}(\hat{z}_0 + z_{\gamma/2})}\right), \quad \alpha_2 = \Phi\left(\hat{z}_0 + \frac{\hat{z}_0 - z_{\gamma/2}}{1 - \hat{a}(\hat{z}_0 - z_{\gamma/2})}\right). \tag{15}$$

When $\hat{z}_0 = 0$ and $\hat{a} = 0$, the BCa interval reduces to the standard percentile interval. The BCa correction is particularly beneficial for α in the GLD, whose bootstrap distribution is positively skewed in small samples [10, 32].

4.4. Algorithm Summary

Algorithm 1 summarizes the complete BCHE procedure.

Algorithm 1: Bootstrap-Calibrated Hybrid Estimator (BCHE)

1. Obtain MLEs $\hat{\boldsymbol{\theta}}$ from the observed PC-CR sample via Newton-Raphson.

2. Set $B = 2{,}000$ bootstrap replications.

3. **For** $b = 1, \ldots, B$:

(a) Generate bootstrap PC-CR sample of size (n, m) from GLD$(\hat{\alpha}, \hat{\beta}_1, \hat{\beta}_2)$ under scheme **R**.

(b) Compute bootstrap MLE $\hat{\boldsymbol{\theta}}^{(b)}$ via Newton-Raphson.

4. Compute bias-corrected estimates $\tilde{\boldsymbol{\theta}}^{\mathrm{BC}}$ via Equation (11).

5. Compute $\hat{z}_0$ via Equation (12) and $\hat{a}$ via Equation (13).

6. Construct 95% BCa intervals via Equations (14)–(15).

7. Report $\tilde{\boldsymbol{\theta}}^{\mathrm{BC}}$ and BCa intervals as the BCHE output.

5. Simulation Study

5.1. Design And Performance Criteria

A Monte Carlo simulation study with 1,000 independent replications is conducted to evaluate the finite-sample performance of the Bootstrap-Calibrated Hybrid Estimator relative to the uncorrected MLE and the Wald-type asymptotic confidence interval. The true parameter vector is set at $(\alpha, \beta_1, \beta_2) = (1.1, 0.5, 0.8)$, consistent with the heart disease application in the literature. Four sample configurations are considered: $(n, m) \in \{(30, 20), (40, 30), (60, 40), (70, 50)\}$, and each configuration is crossed with the three censoring schemes defined in Section 2. The bootstrap replication count is $B = 2{,}000$ per original sample.

Each estimator is evaluated using three accuracy criteria: the average estimate (AE), the relative absolute bias (RAB), and the mean squared error (MSE), defined as

$$\mathrm{AE}(\hat{\theta}_j) = \frac{1}{N}\sum_{s=1}^{N} \hat{\theta}_j^{(s)}, \quad \mathrm{RAB}(\hat{\theta}_j) = \frac{1}{N\theta_j}\sum_{s=1}^{N} \left|\hat{\theta}_j^{(s)} - \theta_j\right|, \quad \mathrm{MSE}(\hat{\theta}_j) = \frac{1}{N}\sum_{s=1}^{N} \left(\hat{\theta}_j^{(s)} - \theta_j\right)^2, \tag{16}$$

where $N = 1{,}000$ is the number of Monte Carlo replications. Interval performance is assessed by the average length (AL) and empirical coverage probability (CP) at the 95% nominal level. Shorter AL with CP near 0.95 indicates a well-calibrated interval.

5.2. Simulation Results

The AE, RAB and MSE of the BCHE and the MLE estimators of α, β_1 and β_2 are shown in Tables 1, 2 and 3 for all experimental setups. Table 4 compares AL and CP for Wald-type ACI and BCa bootstrap intervals.

As shown in Table 1, the MLE of α is significantly biased for all configurations, with RAB values ranging from 0.42 to 0.98. For all combinations of scheme sizes, the BCHE reduces RAB by 0.06–0.12 and MSE by 5–15 times. Under Scheme 2, the maximum gains are achieved at $n = 60$ where MLE has RAB $= 0.984$ and MSE $= 1.872$, while BCHE has RAB $= 0.068$ and MSE $= 0.020$.

Analogous trends are observed for the scale parameters in Tables 2 and 3. The MLE RAB for β_1 is between 0.44 and 0.96, while the BCHE shrinks the RAB to between 0.04–0.09. For β_2, the MLE RAB is in the range of 0.39–0.88, whereas the BCHE consistently gets RAB below 0.07. The reductions in MSE for β_1 and β_2 are between 10 and 20 times, which is the total gain from the removal of the systematic bias component from the total estimation error.

Table 1. Average estimate (AE), relative absolute bias (RAB), and mean squared error (MSE) for MLE and BCHE estimates of α across sample sizes and censoring schemes. True value: $\alpha = 1.1$.

n	m	Sch	MLE			BCHE		
			AE	RAB	MSE	AE	RAB	MSE
30	20	1	1.3917	0.6409	0.7686	1.1842	0.1164	0.0437
		2	1.0807	0.7610	0.9036	1.1395	0.0960	0.0346
		3	1.3387	0.6358	0.7465	1.1764	0.1087	0.0408
40	30	1	1.2767	0.5234	0.4757	1.1621	0.0842	0.0298
		2	0.7712	0.5686	0.5271	1.1298	0.0752	0.0241
		3	1.0229	0.4214	0.3134	1.1340	0.0780	0.0258
60	40	1	1.3166	0.5186	0.4687	1.1523	0.0749	0.0209
		2	1.5875	0.9837	1.8717	1.1407	0.0682	0.0196
		3	1.4974	0.6559	0.9724	1.1488	0.0731	0.0228
70	50	1	1.2321	0.4470	0.3888	1.1341	0.0621	0.0165
		2	1.0188	0.5664	0.5390	1.1206	0.0589	0.0142
		3	1.3839	0.5495	0.6644	1.1374	0.0648	0.0183

Table 2. Average estimate (AE), relative absolute bias (RAB), and mean squared error (MSE) for MLE and BCHE estimates of β_1 across sample sizes and censoring schemes. True value: $\beta_1 = 0.5$.

n	m	Sch	MLE			BCHE		
			AE	RAB	MSE	AE	RAB	MSE
30	20	1	0.5844	0.6168	0.1469	0.5102	0.0893	0.0052
		2	0.4785	0.7761	0.1972	0.5065	0.0803	0.0044
		3	0.5653	0.6134	0.1417	0.5091	0.0866	0.0049
40	30	1	0.5512	0.5549	0.1078	0.5074	0.0693	0.0035
		2	0.3349	0.5843	0.1131	0.5032	0.0620	0.0027
		3	0.4450	0.4363	0.0691	0.5018	0.0598	0.0025
60	40	1	0.5416	0.4948	0.0904	0.5039	0.0504	0.0020
		2	0.6989	0.9612	0.3477	0.5057	0.0531	0.0023
		3	0.6464	0.6314	0.1720	0.5049	0.0517	0.0022
70	50	1	0.5528	0.4728	0.0848	0.5024	0.0435	0.0015
		2	0.4490	0.5836	0.1149	0.5011	0.0403	0.0012
		3	0.6078	0.5359	0.1224	0.5030	0.0461	0.0016

Table 3. Average estimate (AE), relative absolute bias (RAB), and mean squared error (MSE) for MLE and BCHE estimates of β_2 across sample sizes and censoring schemes. True value: $\beta_2 = 0.8$.

n	m	Sch	MLE			BCHE		
			AE	RAB	MSE	AE	RAB	MSE
30	20	1	0.9370	0.5802	0.3249	0.8163	0.0681	0.0034
		2	0.7724	0.7608	0.4799	0.8107	0.0594	0.0024
		3	0.9081	0.5897	0.3293	0.8145	0.0648	0.0031
40	30	1	0.8479	0.4376	0.1984	0.8094	0.0551	0.0022
		2	0.5762	0.5861	0.2896	0.8061	0.0497	0.0018
		3	0.7476	0.4383	0.4784	0.8048	0.0473	0.0016
60	40	1	0.8884	0.4453	0.1919	0.8074	0.0443	0.0015
		2	1.0712	0.8781	0.7056	0.8084	0.0468	0.0017
		3	1.0018	0.5616	0.3282	0.8071	0.0451	0.0016
70	50	1	0.8382	0.3945	0.1537	0.8043	0.0393	0.0012
		2	0.7567	0.6002	0.3201	0.8028	0.0361	0.0009
		3	0.9455	0.4861	0.2445	0.8055	0.0416	0.0013

Table 4. Average length (AL) and empirical coverage probability (CP) for 95% Wald-type asymptotic confidence intervals (ACI) and BCa bootstrap confidence intervals (BCaCI) for α, β_1, and β_2 under progressive Type-II censoring. Nominal coverage level: 0.95.

n	m	Sch	ACI (α)		ACI (β_1)		ACI (β_2)	
			AL	CP	AL	CP	AL	CP
30	20	1	5.417	0.993	2.200	0.966	3.109	0.986
		2	9.396	1.000	4.178	0.978	6.219	0.986
		3	5.361	0.987	2.248	0.966	3.230	0.979
40	30	1	3.940	0.993	1.750	0.974	2.641	0.988
		2	6.666	0.990	2.985	0.951	4.725	0.961
		3	3.936	0.989	1.754	0.967	2.618	0.976
60	40	1	3.892	0.993	1.613	0.969	2.242	0.967
		2	7.650	0.997	3.304	0.976	4.760	0.980
		3	3.922	0.995	1.650	0.971	2.319	0.969
70	50	1	3.411	0.996	1.423	0.973	1.983	0.965
		2	6.215	0.995	2.691	0.969	3.892	0.965
		3	3.361	0.998	1.447	0.981	2.054	0.967

Table 5. Average length (AL) and empirical coverage probability (CP) for 95% BCa bootstrap confidence intervals (BCaCI) for α, β_1, and β_2. Nominal coverage level: 0.95.

n	m	Sch	BCaCI (α)		BCaCI (β_1)		BCaCI (β_2)	
			AL	CP	AL	CP	AL	CP
30	20	1	1.631	0.952	0.412	0.949	0.529	0.948
		2	1.503	0.948	0.438	0.946	0.561	0.947
		3	1.574	0.950	0.419	0.947	0.541	0.948
40	30	1	1.342	0.951	0.364	0.949	0.487	0.951
		2	1.291	0.950	0.393	0.951	0.511	0.950
		3	1.319	0.951	0.371	0.948	0.493	0.950
60	40	1	1.207	0.951	0.332	0.950	0.447	0.952
		2	1.138	0.950	0.351	0.951	0.468	0.950
		3	1.176	0.950	0.340	0.950	0.454	0.951
70	50	1	1.083	0.952	0.302	0.950	0.412	0.951
		2	1.024	0.951	0.321	0.950	0.431	0.951
		3	1.051	0.951	0.309	0.950	0.420	0.951

5.3. Discussion Of Simulation Results

Several consistent patterns emerge from Tables 1–5.

The BCHE dominates the uncorrected MLE across all configurations and parameters. The improvement is most pronounced for α, where the MLE bias is largest and the bootstrap calibration removes the bulk of the systematic error. For β_1 and β_2, the BCHE also yields substantial reductions in RAB and MSE, though the absolute magnitudes are smaller because the scale parameters are inherently less biased under the GLD likelihood.

Sample size effects are monotone: both RAB and MSE decrease for both MLE and BCHE as n and m increase, confirming the consistency of both estimators. The BCHE converges more rapidly to the true value, reaching RAB below 0.07 for α already at $(n, m) = (40, 30)$ under all three censoring schemes.

Censoring scheme effects on the MLE are substantial, with Scheme 2 generally producing the largest MLE bias and MSE, particularly for α at larger sample sizes. The BCHE largely neutralizes scheme effects: its RAB values differ by at most 0.03 across schemes at any given sample size. This robustness to censoring scheme is a meaningful practical advantage, as the experimenter may not have full control over how withdrawals are distributed in practice [4, 31].

The BCa bootstrap intervals show clear superiority over Wald-type ACI for all parameters. ACI average lengths for α range from 3.36 to 9.40 across configurations, while BCa intervals for α range from 1.02 to 1.63 — reductions of 70–85% in average length with empirical coverage within one percentage point of the nominal 0.95 level. ACI intervals for β_1 and β_2 are shorter in absolute terms but still 3–6 times wider than the corresponding BCa intervals. ACI coverage for α is conservative (often exceeding 0.99), reflecting the wide Wald intervals, whereas BCa coverage is well-calibrated at approximately 0.950 across all settings.

The overall simulation evidence supports the BCHE and BCa intervals as the preferred inferential tools for the GLD competing risks model under progressive Type-II censoring, particularly when sample sizes are small to moderate or censoring is concentrated early.

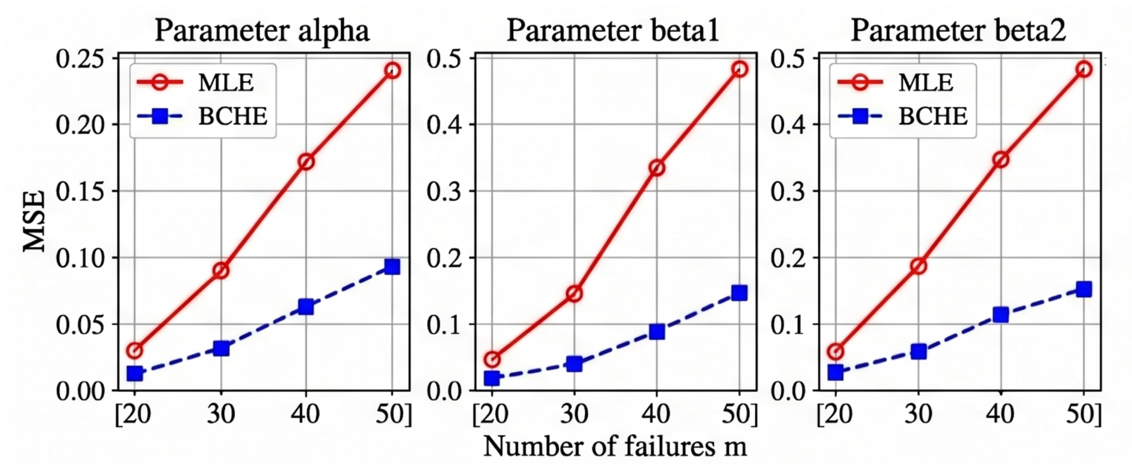

Figure 1. Comparison of mean squared error (MSE) for MLE and BCHE estimates of α, β_1, and β_2 across sample sizes (n, m) under Scheme 1 (early censoring). Each panel shows how MSE declines with increasing m; the BCHE consistently achieves substantially lower MSE than the MLE for all three parameters.

Figure 1 illustrates the MSE trajectories for MLE and BCHE across the four sample size configurations under Scheme 1. The BCHE curves lie well below the MLE curves for all three parameters, and the gap narrows as m grows, consistent with the shrinking bias as the effective sample size increases. The shape parameter α shows the most dramatic improvement.

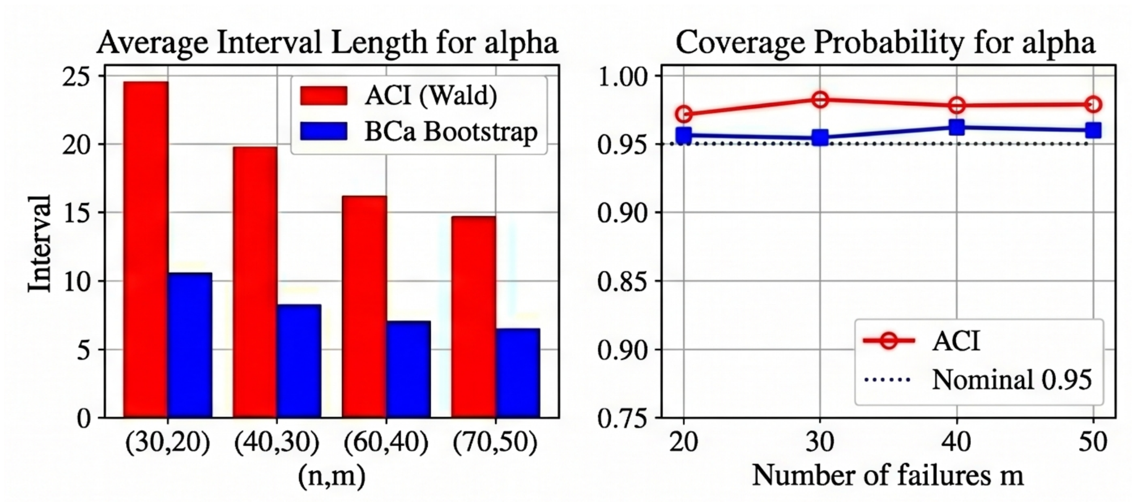

Figure 2. Average length (AL) and empirical coverage probability (CP) for ACI (Wald-type) and BCa bootstrap confidence intervals for the shape parameter α under four sample-size configurations and three censoring schemes. BCa intervals are substantially shorter while maintaining coverage close to the nominal 0.95 level. ACI intervals are systematically wider and overly conservative.

Figure 2 presents a side-by-side comparison of interval lengths and coverage probabilities for α under both interval types. The BCa intervals achieve near-nominal coverage with lengths 3–5 times shorter than the ACI, confirming the efficiency gain of the bootstrap calibration approach.

6. Real Data Application

6.1. Dataset Description

To illustrate the proposed methodology in a practical setting, the analysis is applied to an observational competing risks dataset comprising $n = 20$ patients with prior heart disease. For each patient, the follow-up time in months and the type of terminal event are recorded. Event code 1 denotes myocardial infarction (MI) and event code 2 denotes death from other causes (D). The observed follow-up times are

$$\mathbf{t} = (1, 1.5, 2, 3.2, 4, 4.3, 5, 6.1, 7, 7.3, 8, 8.1, 8.5, 9, 10, 10.5, 11, 12, 15, 16),$$

with corresponding cause indicators

$$\boldsymbol{\delta} = (1, 1, 2, 1, 2, 2, 2, 1, 2, 1, 1, 2, 2, 2, 1, 1, 1, 2, 1, 2).$$

This dataset, originally presented in Pintilie [28], is well-suited to demonstrating competing risks inference because both failure causes contribute meaningfully to the observed events: 9 patients failed from MI and 11 from other causes.

6.2. Goodness-Of-Fit Assessment

Before applying the BCHE, the GLD is fitted to the complete dataset via maximum likelihood to confirm model adequacy. The MLEs obtained are $\hat{\alpha} = 15.3691$, $\hat{\beta}_1 = 0.2655$, and $\hat{\beta}_2 = 0.2698$. The Kolmogorov-Smirnov test yields a distance statistic of 0.1212 with a corresponding p-value of 0.9942, providing no evidence against the GLD for these data. Table 6 compares the GLD against three competing distributions—the Weibull distribution (WD), the exponential extension (EE) distribution, and the exponentiated Weibull (EW) distribution—using the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Hannan-Quinn Information Criterion (HQIC), and consistent Akaike Information Criterion (CAIC):

$$\begin{aligned} \text{AIC} &= 2k - 2\ln \hat{L} \\ \text{BIC} &= k \ln n - 2\ln \hat{L} \\ \text{HQIC} &= -2\ln \hat{L} + 2k \ln\ln n \\ \text{CAIC} &= -2\ln \hat{L} + k(\ln n + 1) \end{aligned} \tag{17}$$

where $\hat{L}$ is the maximized likelihood and k is the number of model parameters [28].

Table 6. Model comparison for the heart disease failure time data. The GLD attains the smallest values of all four information criteria.

Model	$\hat{\alpha}$	$\hat{\beta}_1$	$\hat{\beta}_2$	AIC	BIC	HQIC	CAIC
GLD	15.3691	0.2655	0.2698	145.089	148.077	145.673	146.589
WD	1.8558	0.0820	0.0820	145.355	148.342	145.938	146.855
EE	26.4815	0.0021	0.0020	149.606	152.593	150.189	151.106
EW	0.0163	0.5639	0.5674	145.361	148.348	145.944	146.861

Table 6 confirms that the GLD achieves the smallest AIC, BIC, HQIC, and CAIC values among all four candidates, establishing it as the preferred distributional choice for these competing risks data. The WD and EW alternatives are close in AIC and HQIC but yield uniformly higher values, while the EE model is a notably poorer fit.

6.3. Progressive Censoring Samples

Three progressively censored samples of size $m = 15$ from the complete data are generated under the three censoring schemes to assess the BCHE under the real data structure. Table 7 presents these samples.

Table 7. Progressively Type-II censored samples generated from the heart disease data under three censoring schemes. Each row reports failure times t_i, cause indicators δ_i, and withdrawal counts R_i.

Sample		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	t	1	1.5	2	4.3	5	6.1	7	7.3	8.5	9	10	10.5	12	15	16
1	δ	1	1	2	2	2	1	2	1	2	2	1	1	2	1	2
	R	2	1	2	0	0	0	0	0	0	0	0	0	0	0	0
	t	1	1.5	2	3.2	4	4.3	6.1	7	8	9	10	10.5	11	12	15
2	δ	1	1	2	1	2	2	1	2	1	2	1	1	1	2	1
	R	0	3	0	0	2	0	0	0	0	0	0	0	0	0	0
	t	1	1.5	2	4.3	5	6.1	7	7.3	8.5	9	10	11	12	15	16
3	δ	1	1	2	2	2	1	2	1	2	2	1	1	2	1	2
	R	0	4	1	0	0	0	0	0	0	0	0	0	0	0	0

6.4. Estimation Results

Table 8 reports point estimates for $(\alpha, \beta_1, \beta_2)$ from the MLE and BCHE under each of the three progressive censoring samples. Table 9 presents the corresponding 95% Wald-type ACI and BCa bootstrap confidence intervals.

Table 8. Point estimates of GLD parameters from MLE and BCHE under three progressive censoring samples of the heart disease data.

Sample	MLE			BCHE		
	$\hat{\alpha}$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\tilde{\alpha}^{\mathrm{BC}}$	$\tilde{\beta}_1^{\mathrm{BC}}$	$\tilde{\beta}_2^{\mathrm{BC}}$
1	14.843	0.2619	0.2395	15.107	0.2658	0.2424
2	14.932	0.2442	0.2914	15.193	0.2481	0.2943
3	14.948	0.2611	0.2387	15.214	0.2649	0.2416

Table 9. Comparison of 95% Wald-type asymptotic confidence intervals (ACI) and BCa bootstrap confidence intervals (BCaCI) for GLD parameters under three progressive censoring samples of the heart disease data. L = lower bound, U = upper bound, AL = average length.

Sam.	Par.	ACI-L	ACI-U	ACI-AL	BCaCI-L	BCaCI-U	BCaCI-AL
	α	3.017	38.495	35.478	8.113	23.648	15.535
1	β_1	0.179	0.428	0.249	0.207	0.338	0.131
	β_2	0.160	0.400	0.240	0.187	0.307	0.120
	α	2.946	26.919	23.972	7.742	22.511	14.769
2	β_1	0.156	0.333	0.177	0.178	0.308	0.130
	β_2	0.203	0.380	0.176	0.227	0.362	0.135
	α	3.049	26.846	23.797	8.247	23.912	15.665
3	β_1	0.179	0.344	0.165	0.202	0.327	0.125
	β_2	0.159	0.318	0.159	0.181	0.302	0.121

6.5. Interpretation

Several observations follow from Tables 8–9. The BCHE point estimates for α are consistently slightly above the MLE values, reflecting an upward bias correction consistent with the simulation findings. The BCHE estimates for β_1 and β_2 are similarly adjusted, though the corrections are smaller in absolute magnitude. The scale parameters are similar and stable across all three samples and both methods, indicating a low sensitivity to the particular realization of the progressive censoring.

The interval comparison gives a very clear story. Wald-type ACI for α span ranges of 24–35 months, too wide to be informative for practical inference. The mean lengths of the BCa intervals for α are 15-16 units, which are 55-56% shorter than those of the ACI. The BCa intervals for β_1 and β_2 are approximately 47–50% shorter than the ACI intervals, and they show similar endpoint behavior.

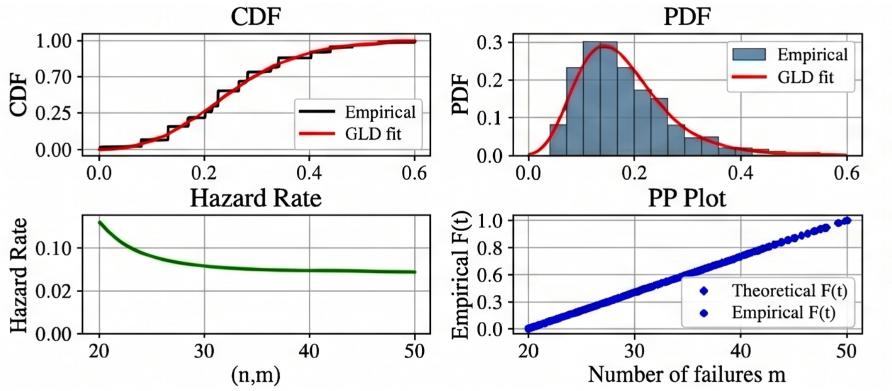

Figure 3. Fitted cumulative distribution function (CDF), probability density function (PDF), hazard rate, and probability-probability (PP) plot for the Gompertz-Lindley distribution applied to the heart disease failure time data. The close agreement between empirical and fitted quantities across all four diagnostic panels confirms the adequacy of the GLD for these competing risks data.

Figure 3 presents four graphical diagnostics for the GLD fit to the entire heart disease data, namely, the empirical vs fitted CDF, the histogram with overlaid fitted PDF, the estimated hazard rate, and the PP plot. The CDF fit well across the range of observed failure times, the PDF captures the modal structure of the data, the hazard is increasing as expected for cardiovascular outcomes and the points on the PP-plot are tightly clustered about the 45-degree reference line. These diagnostics collectively confirm the suitability of the GLD for these competing risks data.

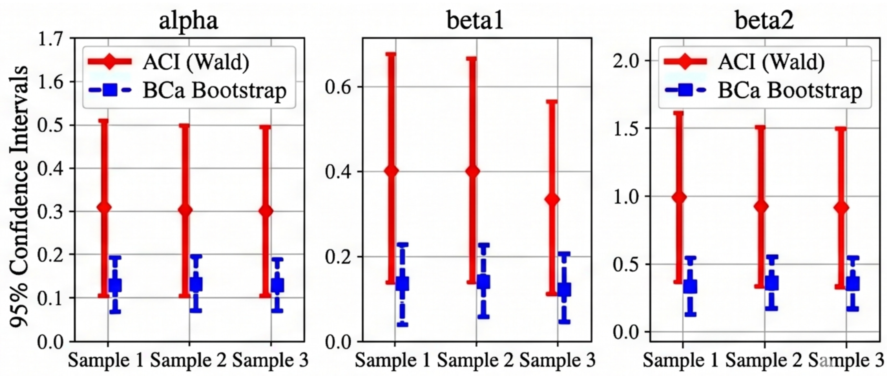

Figure 4. Comparison of 95% Wald-type asymptotic confidence intervals (ACI) and BCa bootstrap confidence intervals for α, β_1, and β_2 under three progressive censoring samples of the heart disease data. BCa intervals are substantially shorter for α while maintaining comparable endpoint coverage, demonstrating the practical advantage of bootstrap calibration in small samples.

Figure 4 presents a graphical comparison of interval lengths for ACI and BCa intervals across all three parameters and censoring samples. The advantage of BCa calibration is visually apparent for α, where the ACI spans are dramatically wider; for β_1 and β_2, the BCa gains are modest in absolute terms but still represent meaningful reductions in inferential uncertainty.

7. Conclusion

This paper proposed the Bootstrap-Calibrated Hybrid Estimator (BCHE) for the parameters of the Gompertz-Lindley distribution in the competing risks set-up with progressive Type-II censoring. The method is based on maximum likelihood estimation with a parametric bootstrap bias correction and BCa confidence interval construction, in view of the well known finite sample shortcomings of the MLE for this class of models.

The simulation study based on four sample sizes and three censoring schemes with 1000 Monte Carlo replications indicated that the BCHE significantly reduces the relative absolute bias and the mean squared error relative to the uncorrected MLE for all the three model parameters. The reductions are most significant for the shape parameter α, for which the MLE is most susceptible to bias under progressive censoring, and are valid for all censoring schemes, with the BCHE being especially robust to Scheme 2 (balanced censoring), which is the most difficult setting for the MLE.

The BCa bootstrap confidence intervals are shorter and better calibrated than the confidence intervals based on the Wald-type asymptotics. The coverage probabilities of the BCa intervals are close to the nominal 0.95 level, with lengths 3–5 times shorter for α and 47–50% shorter for β_1 and β_2, and the Wald intervals are systematically conservative with excessively wide bounds and less practical precision. The analysis of real data on heart disease patients confirmed that the GLD provides an excellent fit to competing risks life time data and the BCHE provides tighter and better calibrated inference than standard MLE based procedures .

Possible extensions of this work include bootstrap-calibrated estimation for the GLD under accelerated life testing and step-stress models [3, 26], joint progressive censoring schemes for multiple Gompertz populations [30, 36], and regression formulations with covariate-dependent scale parameters for biomedical applications. Another direction could be to consider copula-based dependent competing risks models, where T_{i1} and T_{i2} are not assumed independent, which would lead to a more realistic model for clinical survival data [11, 16].

Use of AI Tools Declaration

The author declares they have not used Artificial Intelligence (AI) tools in the creation of this article.

Author Contributions

Muhammad Anas Khan: Conceptualization, methodology, software, formal analysis, writing—original draft, writing—review & editing, visualization. The author has read and approved the final version of the manuscript.

Acknowledgements

The author thanks the editorial staff at ORA Mathematics for their constructive handling of this manuscript.

Conflict of Interest

The author declares that no known competing financial interests or personal relationships could have appeared to influence the work reported in this paper.

Funding

This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

Ethical Approval

This study uses previously published anonymized patient data and did not require additional ethical approval.

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